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# Noether currents and charges for Maxwell-like Lagrangians

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**Abstract**

The Hilbert–Noether theorem states that a current associated with diffeomorphism invariance of a Lagrangian vanishes on shell modulo a divergence of an arbitrary superpotential. Application of the Noether procedure to physical Lagrangians yields, however, meaningful (and measurable) currents. The well-known solution to this 'paradox' is to involve the variation of the metric tensor. Such procedure, for the field considered on a fixed (flat) background, is sophisticated logically (one needs to introduce the variation of a fixed field) and formal. We analyse the Noether procedure for a generic diffeomorphism invariant $p$-form field model. We show that the Noether current of the field considered on a variable background coincides with the current treated in a fixed geometry. Consistent description of the canonical energy–momentum current is possible only if the dynamics of the geometry (gravitation) is taken into account. However, even the 'truncated' consideration yields the proper expression. We examine the examples of the free $p$-form gauge field theory, the GR in the coframe representation and the metric-free electrodynamics. Although the variation of a metric tensor is not acceptable in the latter case, the Noether procedure yields the proper result.

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## 1. Introduction

Probably, the main problem of the Noether procedure is to establish a reasonable correspondence between the set of physical meaningful and measurable currents on the one hand and the set of formal Noether currents on the other hand [1–21]. It is well known that the relation between these two types of quantities is highly non-trivial. In particular, the Noether theorem states that a current associated with a gauge symmetry of a Lagrangian necessarily vanishes on shell modulo a divergence of an arbitrary superpotential. Consequently, this

Noether current is not observable and is physically meaningless. On the other hand, the superpotential (Noether charge) is known to play the crucial role in Wald's analysis of the black-hole entropy [22, 23].

It was recognized long ago that, in order to have a proper description of the electromagnetic current, one has to extend the pure electrodynamic system by introducing charged scalar and/or fermionic fields.

We will show in this paper that the situation for the energy–momentum current is fairly similar. Namely, the consistent description of the energy–momentum current via the Noether procedure is only possible if the gravitational field is taken into account.

The purpose of this paper is to carry out this task for a $p$-form gauge field $\psi$, or, as one also says, for an antisymmetric tensor field [24, 25]. This field can be furnished with some interior indices, i.e., it can be tensorial valued. We assume that the Lagrangian depends only on the field $\psi$ and its exterior derivative $d\psi$. Certainly, our requirements are rather restrictive and take out of the consideration a wide class of mathematically interesting models. The aim of this paper is to examine how the problems of the Noether procedure appear in this relatively simple context. Note that even with these restrictions, our framework is suitable for the description of many interesting physical models, including gravity.

The organization of the paper is as follows. In the next section, neglecting any geometric features of the manifold, we recall the standard Noether procedure for a generic Lagrangian of a $p$-form field. We restate the well-known fact that the Noether current, associated with diffeomorphism invariance, vanishes identically on shell. It should be noted that our treatment is completely local, so diffeomorphism invariance always refers to local diffeomorphism transformations. In particular, we ignore the topological obstructions (such as non-orientability) which can forbid the existence of global frame fields and of global volume forms.

We argue that this triviality of the conserved current appears because the geometry of the manifold is completely ignored in this consideration. A viable Lagrangian density has to be represented by an odd (twisted) differential form. Such a Lagrangian cannot be constructed only out of even forms: the field and its derivative. It always has to involve some odd operator of the type of Hodge's dual map, which changes the parity of the form. Such an odd operator necessarily depends on the geometric features of the manifold, similar to the Hodge map, which depends on the metric tensor.

In order to find out a non-trivial conserved current, we consider in the second section a model for a $p$-form field given on a fixed coframe background. The variation procedure now takes into account, with some restrictions, the variation of the coframe field in addition to the variation of the field $\psi$. The total Noether current of the system vanishes on shell. However, now we are able to identify a non-trivial energy–momentum current for the field $\psi$ with a piece of this trivial total Noether current. The derivation involves, however, some logical contradictoriness. Namely, the coframe field has to be considered as subject to variations, even if it is fixed. The situation is similar to the derivation of the Hilbert energy–momentum tensor for a field given on a fixed manifold. This justifies the necessity to involve a dynamical coframe field, or, in physical language, to consider the gravity field together with the field $\psi$.

In the third section we consider a system of a $p$-form field and a coframe field, both dynamical. We show that the derivative of the Lagrangian with respect to the coframe (Hilbert current) plays a role of the conserved source in the coframe field equation. Again, the total Noether current, associated with the diffeomorphism invariance of the system, vanishes on shell. However, this equation now represents a relation between the Noether and the Hilbert currents. Moreover, we derive in this way an explicit expression of the Hilbert current, of the Noether charge, and the Noether identity. All these quantities are well defined.

In the fourth section we deal with a generic first-order Lagrangian shifted by a total derivative. We show that such shift preserves the field equations as well as the conserved energy–momentum current. The Noether charge is shifted, however.

In the fifth section we consider three examples of generalized Maxwell Lagrangians. We examine the free $p$-form gauge field theory, the GR in the coframe representation and the metric-free electrodynamics.

## 2. A *p*-form Lagrangian

### *2.1. Non-geometric Lagrangian*

Consider a $p$-form field $\psi$ defined on an $n$-dimensional differential manifold $M$. It is a straightforward generalization of the ordinary 1-form potential field $A$ of 4D Maxwell electrodynamics. We describe the dynamics of the field $\psi$ by a generic Lagrangian $n$-form of first order, i.e., only the field and its exterior (first-order) derivative are involved,

$$\mathcal{L} = \mathcal{L}(\psi, \mathrm{d}\psi) \tag{2.1}$$

see, for instance, [1]. Denote the derivatives of the Lagrangian taken with respect to the field $\psi$ by

$$\sigma := \frac{\partial \mathcal{L}}{\partial \psi} \qquad \pi := \frac{\partial \mathcal{L}}{\partial (\mathrm{d}\psi)}. \tag{2.2}$$

We will refer to the $(n-p)$-form $\sigma$ as the *current of the field* $\psi$, and to the $(n-p-1)$-form $\pi$ as the *field strength*.

Using these abbreviations, the variation of the Lagrangian (2.1), in the exterior form notation, may be written as

$$\delta\mathcal{L} = \delta\psi \wedge \sigma + \delta(\mathrm{d}\psi) \wedge \pi. \tag{2.3}$$

Applying the commutativity of the operators d and $\delta$, we extract the total derivative and obtain the variational relation

$$\delta\mathcal{L} = \delta\psi \wedge \mathcal{E} + \mathrm{d}\Omega \tag{2.4}$$

where the $(n-p)$-form

$$\mathcal{E} := \sigma - (-1)^p\, \mathrm{d}\pi \tag{2.5}$$

is the action of the Euler–Lagrange operator on the field $\psi$. The $(n-1)$-form $\Omega$ is defined as

$$\Omega := \delta\psi \wedge \pi. \tag{2.6}$$

The form $\Omega$ is linear in variations of the field. This quantity is sometimes referred to as the *pre-simplectic potential* [22, 23]. Observe that for a given Lagrangian both quantities $\mathcal{E}$ and $\Omega$ are well defined without any ambiguity.

Consider the variations of the fields which vanish at a boundary of a region. We obtain the field equation which immediately yields the conservation law for the $(n-p)$-form $\sigma$:

$$\mathcal{E} = 0 \quad \text{or} \quad \mathrm{d}\pi = (-1)^p\sigma \quad \Longrightarrow \quad \mathrm{d}\sigma = 0. \tag{2.7}$$

Returning to the variational relation (2.4), we recognize a special case, when the variation of the Lagrangian is an exact form $\delta\mathcal{L} = \mathrm{d}S$. Equation (2.4) now implies the existence of an $(n-1)$-form $\Theta := -S + \Omega$ which is conserved modulo the field equations, i.e., on shell:

$$\mathrm{d}\Theta + \delta\psi \wedge \mathcal{E} = 0 \quad \Longleftrightarrow \quad \mathrm{d}\Theta \approx 0. \tag{2.8}$$

Here and in the following, we use the symbol '$\approx$' for 'equal up to a linear combination of the field equation form $\mathcal{E}$'.

The Lagrangian (2.1) depends only on the exterior form field and its exterior derivative, thus it is diffeomorphism invariant. In order to specialize the diffeomorphism invariance, the variations of the field have to be generated by the Lie derivative taken with respect to a smooth vector field $\xi$,

$$\delta\psi = L_\xi \psi = \mathrm{d}(\xi\rfloor\psi) + \xi\rfloor\mathrm{d}\psi. \tag{2.9}$$

Because of the diffeomorphism invariance, the variation of the $n$-form Lagrangian is induced by the Lie derivative taken with respect to the same vector field $\xi$, i.e.,

$$\delta\mathcal{L} = L_\xi \psi = \mathrm{d}(\xi\rfloor\mathcal{L}). \tag{2.10}$$

Accordingly, we have an $(n-1)$-form

$$\Theta(\xi) = -\xi\rfloor\mathcal{L} + \Omega(\xi) \tag{2.11}$$

generated by the diffeomorphism symmetry of the Lagrangian, which is weak (on shell) conserved

$$\mathrm{d}\Theta(\xi) + \delta\psi \wedge \mathcal{E} = 0 \qquad \mathrm{d}\Theta(\xi) \approx 0. \tag{2.12}$$

We will refer to $\Theta(\xi)$ as the *Noether current*. The explicit expression for this quantity is derived from (2.9)–(2.11) as

$$\Theta(\xi) = -\xi\rfloor\mathcal{L} + [(\xi\rfloor\mathrm{d}\psi) + \mathrm{d}(\xi\rfloor\psi)] \wedge \pi. \tag{2.13}$$

Because the Lagrangian depends only on $\psi$ and $\mathrm{d}\psi$, also the field strength $\pi$ depends only on these variables. Thus, the Noether current is locally constructed only out of the quantities $\psi$, $\mathrm{d}\psi$, and the undefined vector field $\xi$. Extracting the total derivative in (2.13) and using the field equation (2.7), we decompose the Noether current as

$$\Theta(\xi) = \mathcal{S}(\xi) + \mathrm{d}Q(\xi) \tag{2.14}$$

where

$$\mathcal{S}(\xi) := -\xi\rfloor\mathcal{L} + (\xi\rfloor\mathrm{d}\psi) \wedge \pi + (\xi\rfloor\psi) \wedge \sigma \tag{2.15}$$

is an $(n-1)$-form current, whereas

$$Q(\xi) := (\xi\rfloor\psi) \wedge \pi \tag{2.16}$$

is an $(n-2)$-form charge. Certainly, two currents are conserved on shell simultaneously,

$$\mathrm{d}\Theta(\xi) \approx 0 \quad \Longleftrightarrow \quad \mathrm{d}\mathcal{S}(\xi) \approx 0 \tag{2.17}$$

for an arbitrary vector field $\xi$.

As is proved in [8], a decomposition of a type (2.14) may be provided for an arbitrary diffeomorphism invariant Lagrangian. It means that a total derivative can be extracted from the Noether current in such a way that the remaining term $\mathcal{S}(\xi)$ is (algebraically) linear in the undetermined vector field $\xi$. Decompose $\xi$ into its components according to $\xi = \xi^a e_a$. The current $\mathcal{S}(\xi)$ involves the undefined vector field $\xi$ only in a linear algebraic form. Hence it fulfils

$$\mathcal{S}(\xi) = \xi^a \mathcal{S}(e_a). \tag{2.18}$$

Thus the conservation law for this current reads

$$\mathrm{d}\mathcal{S}(\xi) = \mathrm{d}\xi^a \wedge \mathcal{S}(e_a) + \xi^a\, \mathrm{d}\mathcal{S}(e_a) \approx 0. \tag{2.19}$$

The arbitrariness of $\xi$ means independence of the quantities $\xi^a$ and $\mathrm{d}\xi^a$. Hence, two terms on the right-hand side of (2.19) should vanish simultaneously. Thus we obtain two, so-called, *cascade equations* [1, 29]

$$\xi^a: \qquad d\mathcal{S}(e_a) \approx 0 \tag{2.20}$$

$$d\xi^a: \qquad \mathcal{S}(e_a) \approx 0. \tag{2.21}$$

Observe that (2.20) is not merely a consequence of equation (2.21). Indeed, (2.21) means $\mathcal{S}(e_a) = A_a{}^b \wedge \mathcal{E}_b$, for some $(p-1)$-form $A_a{}^b$. Thus its exterior derivative is not, in general, proportional to the field equation.

Because of (2.18), (2.21), the current $\mathcal{S}(\xi)$ vanishes on shell, i.e., on all solutions of the field equation, for an arbitrary vector field $\xi$,

$$\mathcal{S}(\xi) \approx 0. \tag{2.22}$$

Inserting (2.22) into (2.14) we obtain

$$\Theta(\xi) \approx dQ(\xi). \tag{2.23}$$

In this way, we derive the well-known property of gauge conserved currents (Noether–Hilbert theorem). The conserved Noether current $\Theta(\xi)$, which corresponds to the gauge invariance of the Lagrangian (diffeomorphism in our case), is exact on shell. Although some technical differences, our differential form consideration is similar to the tensorial treatment in [1].

Let us take into account the intermediary result (2.22) of our derivation: the vanishing of the current $\mathcal{S}(\xi)$ on shell. In the case of the ordinary Maxwell field with $p = 1$, $\psi = A$, and

$$\mathcal{L} = -\tfrac{1}{2}\, dA \wedge * dA. \tag{2.24}$$

Equation (2.15) takes the form

$$\mathcal{S}(e_a) = -\tfrac{1}{2} e_a \rfloor (dA \wedge * dA) + (e_a \rfloor dA) \wedge * dA). \tag{2.25}$$

Thus, it is completely identical to the electrodynamic energy–momentum current. This current uniquely defines the ordinary measurable energy–momentum tensor [34]. Thus $\mathcal{S}(\xi)$ cannot vanish for the electrodynamic fields identically. Consequently, at least in the case of the free Maxwell field, we seem to reach a ‘contradiction’ to the Noether–Hilbert theorem.

It is well known, that the problem comes from elimination of the geometric variables from the variation procedure. The consideration above is, in fact, restricted to Lagrangians which depend only on the fields and their derivatives. Such Lagrangians are even (untwisted) forms, thus it is rather natural that they do not contribute to the measurable energy–momentum quantities. Dicke [27] proved that it is almost impossible to construct a non-trivial Lagrangian for a field 'interacted only with itself'. Only for the metric field a non-trivial Lagrangian can be constructed (Hilbert–Einstein). All other physical Lagrangians, in fact, represent interaction with some other field (the metric field in most cases). This is a situation appeared in (2.24), where the metric is involved implicitly by the Hodge operator, which makes the Lagrangian an odd form.

### 2.2. *Non-dynamical coframe*

A non-trivial Lagrangian has to be represented by an odd (twisted) *n*-form. We assume the field $\psi$ to be even (untwisted). Thus, in addition to the even form $\psi$ and its derivatives, the Lagrangian has to include some odd operator on forms. This odd operator necessarily inherits certain geometrical properties of the manifold. It can be, for instance, the ordinary Hodge dual map or the constitutive tensor of the metric-free electrodynamics [34]. Such operator may be defined by the metric tensor or by the coframe field. For the time being, we do not specify the odd operator used and only assume that the Lagrangian depends also on a fixed coframe field $\vartheta^a$,

$$\mathcal{L} = \mathcal{L}(\psi, d\psi, \vartheta^a). \tag{2.26}$$

Certainly, we require the coframe field to be essentially involved in the $p$-form field Lagrangian. It means that the derivatives $\partial\mathcal{L}/\partial\vartheta^a$ and $\partial\mathcal{L}/\partial(d\psi)$ are assumed to be non-zero functions of $d\psi$ and $\vartheta^a$.

The variation of this Lagrangian, taken with respect to $\psi$ and $\vartheta^a$, is

$$\begin{aligned}\delta\mathcal{L} &= \delta\psi \wedge \sigma + \delta d(\psi) \wedge \pi + \delta\vartheta^a \wedge \frac{\partial\mathcal{L}}{\partial\vartheta^a} \\ &= \delta\psi \wedge \mathcal{E} + \delta\vartheta^a \wedge \frac{\partial\mathcal{L}}{\partial\vartheta^a} + d(\delta\psi \wedge \pi).\end{aligned} \tag{2.27}$$

In the absence of constrains, the general variation procedure requires us to consider independent variations of all fields involved in the Lagrangian. Thus in addition to the field equation

$$\mathcal{E} = 0 \quad \Longleftrightarrow \quad d\pi = (-1)^p\sigma \tag{2.28}$$

we obtain $\partial\mathcal{L}/\partial\vartheta^a = 0$. It means that the field $\vartheta^a$ cannot be incorporated into the Lagrangian, at all. In order to overcome this obstacle we require only the variation $\delta\psi$ to be free. It means that the coframe field is considered to be non-dynamical. Thus we have only one field equation (2.28). The variation of the Lagrangian on shell remains in the form

$$\delta\mathcal{L} \approx \delta\vartheta^a \wedge \frac{\partial\mathcal{L}}{\partial\vartheta^a} + d(\delta\psi \wedge \pi). \tag{2.29}$$

Observe that, in contrast to (2.4), the right-hand side of (2.29) is not a total derivative. We may apply, however, the diffeomorphism invariance of the Lagrangian also in this case. Consider again the variation of the fields to be produced by the Lie derivatives. In accordance with the non-dynamical nature of the coframe field, we will require $d\vartheta^a = 0$. Thus relation (2.29) takes the form

$$d\Theta(\xi) + d(\xi\rfloor\vartheta^a) \wedge \frac{\partial\mathcal{L}}{\partial\vartheta^a} \approx 0 \tag{2.30}$$

where $\Theta(\xi)$ is defined in (2.13). We use (2.14) to obtain

$$d\mathcal{S}(\xi) + d(\xi\rfloor\vartheta^a) \wedge \frac{\partial\mathcal{L}}{\partial\vartheta^a} \approx 0 \tag{2.31}$$

where $\mathcal{S}(\xi)$ is defined in (2.15). This equation has to be satisfied for an arbitrary vector field $\xi$. Spelling out (2.30) explicitly for $\xi = \xi^a e_a$, we obtain

$$d\xi^a \wedge \left[\mathcal{S}(e_a) + \frac{\partial\mathcal{L}}{\partial\vartheta^a}\right] + \xi^a d\mathcal{S}(e_a) \approx 0. \tag{2.32}$$

Independence of the quantities $\xi^a$ and $d\xi^a$ yields two cascade equations

$$\xi^a : \qquad d\mathcal{S}(e_a) \approx 0 \tag{2.33}$$

$$d\xi^a : \qquad \mathcal{S}(e_a) \approx -\frac{\partial\mathcal{L}}{\partial\vartheta^a}. \tag{2.34}$$

Thus we resolve the contradiction mentioned above. The first cascade equation represents the weak conservation law for the current $\mathcal{S}(e_a)$. This current does not vanish now, in contrast to (2.21).

In the tensorial approach of field theory, the derivative $\partial\mathcal{L}/\partial g_{\mu\nu}$ represents the Hilbert energy–momentum tensor. In the coframe approach the similar meaning may be given to the derivative $\partial\mathcal{L}/\partial\vartheta^a$. Consequently, the second cascade equation represents the equality between the *canonical current* $\mathcal{S}(e_a)$ and the *coframe Hilbert current* $\partial\mathcal{L}/\partial\vartheta^a$.

The price of this result is some non-completeness of the variation procedure. (i) We were forced to consider the coframe field as fixed and non-dynamical, however we have to take the variation of the Lagrangian also with respect to this field. (ii) The condition $\mathrm{d}\vartheta^a = 0$ was applied, thus we restricted ourself to consider only holonomic coframes.

In order to resolve these problems, we have to make the coframe field dynamical.

## 3. Matter–coframe system

### *3.1. Lagrangian and field equations*

Let us take an $n$-dimensional, smooth, orientable, differential manifold $M$. We describe the geometry on $M$ by a smooth coframe field $\vartheta^a$ and its dual: a frame field $e_a$, where $a = 1, \ldots, n$. The duality is expressed by the relation $e_a \rfloor \vartheta^b = \delta^b_a$, where $\rfloor$ is the interior product operator. The coframe field $\vartheta^a$ is a set of $n$ even (untwisted) 1-forms, which are linear independent at every point of $M$. The duality relation provides the linear independence also of the frame field $e_a$ (a set of $n$ even vector fields). In the case of the teleparallel (coframe) approach to gravity, the manifold $M$ is endowed also with a metric $g = \eta_{ab}\vartheta^a \otimes \vartheta^b$. We will not use time being the metric tensor, thus we are working on a metric-free background endowed with a coframe field.

Assume the matter to be represented by a $p$-form field $\psi$. Certainly, a viable matter system has to also include some set of fermionic fields, which we exclude from the consideration, for the sake of simplicity. Thus we are dealing with some generalization of the Maxwell–Einstein system.

We assume the fields $\vartheta^a$, $e_a$ and $\psi$ to be even (untwisted). It means that they are invariant under a change of orientation of the manifold. We describe the matter–coframe system $\{\vartheta^a, \psi\}$ by a generic Lagrangian form of first order:

$$\mathcal{L} = \mathcal{L}(\psi, \mathrm{d}\psi, \vartheta^a, \mathrm{d}\vartheta^a). \tag{3.1}$$

A non-trivial Lagrangian has to be represented by an odd (twisted) $n$-form. Thus, in addition to the even forms $\psi$, $\vartheta^a$ and their derivatives, the Lagrangian has to involve some odd operator on forms. We do not specify the odd operator and only assume that this operator can be expressed in terms of the coframe field. Denote the derivatives taken with respect to the coframe field as

$$\Sigma_a := \frac{\partial\mathcal{L}}{\partial\vartheta^a} \qquad \Pi_a := \frac{\partial\mathcal{L}}{\partial(\mathrm{d}\vartheta^a)}. \tag{3.2}$$

The odd $(n-1)$-form $\Sigma_a$ will be referred to as the *current of the coframe field* while the odd $(n-2)$-form $\Pi_a$ as the *strength of the coframe field*. The derivatives of the Lagrangian taken with respect to the matter field are defined in (2.2). We will refer now to the odd $(n-p)$-form $\sigma$ as the *current of the matter field*, and to the odd $(n-p-1)$-form $\pi$ as the *strength of the matter field*.

Using the abbreviations (2.2) and (3.2), variation of the Lagrangian may be written as

$$\delta\mathcal{L} = \delta\psi \wedge \sigma + \delta(\mathrm{d}\psi) \wedge \pi + \delta\vartheta^a \wedge \Sigma_a + \delta(\mathrm{d}\vartheta^a) \wedge \Pi_a. \tag{3.3}$$

Extracting the total derivatives, we obtain the variational relation

$$\delta\mathcal{L} = \delta\psi \wedge {}^{(\mathrm{mat})}\mathcal{E} + \delta\vartheta^a \wedge {}^{(\mathrm{gr})}\mathcal{E}_a + \mathrm{d}\Omega \tag{3.4}$$

where the field equation forms are

$${}^{(\mathrm{mat})}\mathcal{E} := \sigma + (-1)^{p+1}\,\mathrm{d}\pi \tag{3.5}$$

$$^{(\mathrm{gr})}\mathcal{E}_a := \Sigma_a + \mathrm{d}\Pi_a \tag{3.6}$$

while the pre-simplectic potential is

$$\Omega := \delta\psi \wedge \pi + \delta\vartheta^a \wedge \Pi_a. \tag{3.7}$$

Observe that for a given Lagrangian all the quantities $^{(\mathrm{mat})}\mathcal{E}$, $^{(\mathrm{gr})}\mathcal{E}_a$ and $\Omega$ are well defined without any ambiguity.

Consider the variations of the fields which vanish at a boundary of a region. We obtain the matter field equation $^{(\mathrm{mat})}\mathcal{E} = 0$, or, explicitly,

$$\mathrm{d}\pi = (-1)^p \sigma \tag{3.8}$$

and the coframe field equation $^{(\mathrm{gr})}\mathcal{E}_a = 0$, i.e.,

$$\mathrm{d}\Pi_a = -\Sigma_a. \tag{3.9}$$

The left-hand sides of equations (3.8), (3.9) are the derivatives of the strengths. Hence, the right-hand sides of these equations represent the sources of the matter field and of the coframe (gravity) field, respectively.

These field equations yield two conservation laws for the sources:

(i) conservation of the matter current

$$\mathrm{d}\sigma = 0 \tag{3.10}$$

(ii) conservation of the coframe current

$$\mathrm{d}\Sigma_a = 0. \tag{3.11}$$

For the generic Lagrangian used, these two conserved currents depend on both fields: matter field $\psi$ and the coframe (gravity) field $\vartheta^a$, i.e., the currents include the contributions of two fields as well as the interaction between them. The conservation laws (3.10), (3.11) are consequences of the field equations. They, however, are strong conservation laws, because their right-hand sides do not involve the combinations of the field equations $^{(\mathrm{mat})}\mathcal{E}$ and $^{(\mathrm{gr})}\mathcal{E}_a$. This is in contrast to the weak Noether currents that will appear below.

### *3.2. Noether current and charge*

In the case that the variation of the Lagrangian is an exact form, $\delta\mathcal{L} = \mathrm{d}S$, the variational relation (3.4) again implies the existence of an $(n-1)$-form $\Theta := S - \Omega$, which is conserved modulo the field equations:

$$\mathrm{d}\Theta + \delta\psi \wedge {}^{(\mathrm{mat})}\mathcal{E} + \delta\vartheta^a \wedge {}^{(\mathrm{gr})}\mathcal{E}_a = 0 \qquad \Longleftrightarrow \qquad \mathrm{d}\Theta \approx 0. \tag{3.12}$$

Consider the variations of the field that are generated by the Lie derivative taken with respect to a smooth vector field $\xi$, i.e.,

$$\delta\psi = L_\xi \psi = \mathrm{d}(\xi \rfloor \psi) + \xi \rfloor \mathrm{d}\psi \tag{3.13}$$

and

$$\delta\vartheta^a = L_\xi \vartheta^a = \mathrm{d}(\xi \rfloor \vartheta^a) + \xi \rfloor \mathrm{d}\vartheta^a. \tag{3.14}$$

The diffeomorphism invariance of the $n$-form Lagrangian yields

$$\delta\mathcal{L} = L_\xi \psi = \mathrm{d}(\xi \rfloor \mathcal{L}). \tag{3.15}$$

Accordingly, we have a weak (on shell) conserved odd $(n-1)$-form generated by the diffeomorphism symmetry of the Lagrangian

$$\Theta(\xi) = -\xi \rfloor \mathcal{L} + \Omega \qquad \mathrm{d}\Theta(\xi) \approx 0. \tag{3.16}$$

We will refer to $\Theta(\xi)$ as the *total Noether current* of the matter–coframe system. The explicit expression for this quantity is derived from (3.7), (3.13), (3.14) as

$$\Theta(\xi) = -\xi\rfloor\mathcal{L} + [(\xi\rfloor d\psi) + d(\xi\rfloor\psi)] \wedge \pi + [(\xi\rfloor d\vartheta^a) + d(\xi\rfloor\vartheta^a)] \wedge \Pi_a. \qquad (3.17)$$

This conserved current is locally constructed out of the fields appearing in the Lagrangian and of the unspecified vector field $\xi$.

Extracting the total derivatives and applying the field equations, we decompose this current as

$$\Theta(\xi) = \mathcal{S}(\xi) + dQ(\xi) \qquad (3.18)$$

where

$$\mathcal{S}(\xi) = -\xi\rfloor\mathcal{L} + (\xi\rfloor d\psi) \wedge \pi + (\xi\rfloor\psi) \wedge \sigma + (\xi\rfloor d\vartheta^a) \wedge \Pi_a + (\xi\rfloor\vartheta^a) \wedge \Sigma_a \qquad (3.19)$$

whereas

$$Q(\xi) := (\xi\rfloor\vartheta^a) \wedge \Pi_a + (\xi\rfloor\psi) \wedge \pi. \qquad (3.20)$$

The currents $\Theta(\xi)$ and $\mathcal{S}(\xi)$ are weak conserved simultaneously. The current $\mathcal{S}(\xi)$ is algebraically linear in the vector field $\xi$, so $\mathcal{S}(\xi^a e_a) = \xi^a\mathcal{S}(e_a)$. Thus the cascade equations read

$$\xi^a : \qquad d\mathcal{S}(e_a) \approx 0 \qquad (3.21)$$

$$d\xi^a : \qquad \mathcal{S}(e_a) \approx 0. \qquad (3.22)$$

We rewrite them explicitly as

$$\mathcal{S}(e_a) = -e_a\rfloor\mathcal{L} + (e_a\rfloor d\psi) \wedge \pi + (e_a\rfloor\psi) \wedge \sigma + (e_a\rfloor d\vartheta^a) \wedge \Pi_a + \Sigma_a \approx 0. \qquad (3.23)$$

Thus we derive

$$\Sigma_a \approx e_a\rfloor\mathcal{L} - (e_a\rfloor d\psi) \wedge \pi - (e_a\rfloor\psi) \wedge \sigma - (e_a\rfloor d\vartheta^a) \wedge \Pi_a \qquad (3.24)$$

which is the proper conserved and non-trivial energy–momentum current of the system.

Substituting (3.21) into (3.18) we obtain

$$\Theta(\xi) \approx dQ(\xi) \qquad (3.25)$$

where the explicit form of the Noether charge is given in (3.20). This $(n-2)$-form is locally constructed out of the fields appearing in the Lagrangian and $\xi$. For a proof that this is possible in a general diffeomorphism invariant case see [9].

### 3.3. Noether identity

Return to the variational relation (3.12) and consider the case when the variation of the Lagrangian is an exact form. Equation (3.12) can be viewed as a condition that the equation forms $\mathcal{E}$ have to fulfil in order to yield an exact form

$$\delta\psi \wedge {}^{(\text{mat})}\mathcal{E} + \delta\vartheta^a \wedge {}^{(\text{gr})}\mathcal{E}_a \qquad \text{exact}. \qquad (3.26)$$

In the case of diffeomorphism invariance, the first term reads

$$(\xi\rfloor d\psi) \wedge {}^{(\text{mat})}\mathcal{E} - (-1)^p(\xi\rfloor\psi) \wedge d\sigma \qquad (3.27)$$

up to a total derivative. Analogously, the second term in (3.26) gives

$$(\xi\rfloor d\vartheta^a) \wedge {}^{(\text{gr})}\mathcal{E}_a + (\xi\rfloor\vartheta^a) \wedge d\Sigma_a. \qquad (3.28)$$

The sum of the terms (3.27) and (3.28) should be an exact form for an arbitrary $\xi$. Observe, however, that if it is true for some vector field $\xi$ it will not be true for a vector field $f\xi$, where $f$ is an arbitrary function. The only possibility is to require the sum of the terms (3.27) and (3.28) to be zero. Thus we have

$$(\xi\rfloor\vartheta^a)\wedge \mathrm{d}\Sigma_a = (\xi\rfloor \mathrm{d}\psi)\wedge {}^{(\mathrm{mat})}\mathcal{E} - (-1)^p(\xi\rfloor\psi)\wedge \mathrm{d}\sigma - (\xi\rfloor \mathrm{d}\vartheta^a)\wedge {}^{(\mathrm{gr})}\mathcal{E}_a. \tag{3.29}$$

On shell it means

$$(\xi\rfloor\vartheta^a)\wedge \mathrm{d}\Sigma_a \approx (-1)^{p+1}(\xi\rfloor\psi)\wedge \mathrm{d}\sigma. \tag{3.30}$$

We replace the vector field by the vector basis $\xi\rightarrow e_a$ and obtain the Noether identity

$$\mathrm{d}\Sigma_a = (e_a\rfloor \mathrm{d}\psi)\wedge {}^{(\mathrm{mat})}\mathcal{E} - (-1)^p(e_a\rfloor\psi)\wedge \mathrm{d}\sigma - (e_a\rfloor \mathrm{d}\vartheta^b)\wedge {}^{(\mathrm{gr})}\mathcal{E}_b \tag{3.31}$$

or on shell

$$\mathrm{d}\Sigma_a \approx (-1)^{p+1}(e_a\rfloor\psi)\wedge \mathrm{d}\sigma. \tag{3.32}$$

This identity shows that, on shell, two currents, $\Sigma_a$ and $\sigma$ are conserved simultaneously.

## 4. A total derivative in Lagrangians

The form (3.1) of the Lagrangian is not general enough to include all viable Lagrangians. Particularly, the Hilbert–Einstein Lagrangian for gravity involves the second-order derivatives of the metric tensor. The remarkable feature is that the second derivative terms appear in the form of a total derivative. We utilize this property and consider a generic Lagrangian shifted by a total derivative

$$\tilde{\mathcal{L}} = \mathcal{L}(\psi, \mathrm{d}\psi, \vartheta^a, \mathrm{d}\vartheta^a) + \mathrm{d}\Lambda(\psi, \mathrm{d}\psi, \vartheta^a, \mathrm{d}\vartheta^a) \tag{4.1}$$

where $\Lambda$ is an arbitrary $(n-1)$-form locally constructed from the fields and their first-order derivatives only. The total derivative shift, as it is well known, preserves the field equations. Let us examine how the shift (4.1) influences the conserved currents. Because the Lagrangian (4.1) involves second derivatives of the dynamical fields, it is of second order due to the usual classification. However, because the variation operator commutes with the exterior derivative, the first-order formalism is applicable also in this case. Variation of the transformed Lagrangian (4.1) takes the form

$$\delta\tilde{\mathcal{L}} = \delta\mathcal{L} + \mathrm{d}(\delta\Lambda). \tag{4.2}$$

The shift form $\Lambda$ generates additional terms, which may be collected in

$$\delta\tilde{\mathcal{L}} = \delta\psi\wedge\tilde{\sigma} + \delta(\mathrm{d}\psi)\wedge\tilde{\pi} + \delta\vartheta^a\wedge\tilde{\Sigma}_a + \delta(\mathrm{d}\vartheta^a)\wedge\tilde{\Pi}_a \tag{4.3}$$

where the shifted quantities are defined as [21]

$$\tilde{\sigma} := \sigma + (-1)^p \mathrm{d}\left(\frac{\partial\Lambda}{\partial\psi}\right) \tag{4.4}$$

$$\tilde{\pi} := \pi + \frac{\partial\Lambda}{\partial\psi} + (-1)^{p+1}\mathrm{d}\left(\frac{\partial\Lambda}{\partial(\mathrm{d}\psi)}\right) \tag{4.5}$$

and

$$\tilde{\Sigma}_a := \Sigma_a + \mathrm{d}\left(\frac{\partial\Lambda}{\partial\vartheta^a}\right) \tag{4.6}$$

$$\tilde{\Pi}_a := \Pi_a - \frac{\partial\Lambda}{\partial\vartheta_a} - \mathrm{d}\left(\frac{\partial\Lambda}{\partial(\mathrm{d}\vartheta_a)}\right). \tag{4.7}$$

We extract the total derivatives in (4.3) and obtain

$$\delta\tilde{\mathcal{L}} = \delta\psi\wedge {}^{(\mathrm{mat})}\tilde{\mathcal{E}} + \delta\vartheta^a\wedge {}^{(\mathrm{gr})}\tilde{\mathcal{E}}_a + \mathrm{d}(\tilde{\Omega}) \tag{4.8}$$

where the field equations forms are

$$ {}^{(\mathrm{mat})}\tilde{\mathcal{E}} := \tilde{\sigma} - (-1)^p \mathrm{d}\tilde{\pi} \tag{4.9} $$

$$ {}^{(\mathrm{gr})}\tilde{\mathcal{E}}_a := \tilde{\Sigma}_a + \mathrm{d}\tilde{\Pi}_a \tag{4.10} $$

and the pre-simplectic potential is

$$ \tilde{\Omega} := \delta\psi \wedge \tilde{\pi} + \delta\vartheta^a \wedge \tilde{\Pi}_a. \tag{4.11} $$

The corresponding field equations have the same form as (3.8), (3.9). Moreover, because of (4.4)–(4.7), they are equivalent

$$ \mathrm{d}\tilde{\pi} = (-1)^p \tilde{\sigma} \quad \Longleftrightarrow \quad \mathrm{d}\pi = (-1)^p \sigma \tag{4.12} $$

and

$$ \mathrm{d}\tilde{\Pi}_a = (-1)^p \tilde{\Sigma}_a \quad \Longleftrightarrow \quad \mathrm{d}\Pi_a = (-1)^p \Sigma_a. \tag{4.13} $$

The shifted quantities $\tilde{\pi}$ and $\tilde{\Pi}_a$ do not pick up an additional exact form. The conserved currents $\tilde{\sigma}$ and $\tilde{\Sigma}_a$ are shifted only by a total derivative. Thus we are confronted with the known problem of an ambiguity of the conserved current [22]. Two currents $\Sigma_a$ and $\tilde{\Sigma}_a$ are the source terms of the field strengths. They also conserved simultaneously and yield the same value if integrated over a closed surface. Their actual values, however, are different.

Consider the variation relation (4.2). It can be written as

$$ \delta\tilde{\mathcal{L}} = \delta\psi \wedge {}^{(\mathrm{mat})}\mathcal{E} + \delta\vartheta^a \wedge {}^{(\mathrm{gr})}\mathcal{E}_a + \mathrm{d}(\Omega + \delta\Lambda) \tag{4.14} $$

or, on shell,

$$ \delta\tilde{\mathcal{L}} \approx \mathrm{d}(\Omega + \delta\Lambda). \tag{4.15} $$

Thus the diffeomorphism invariance generates a weak conserved current

$$ \begin{aligned} \tilde{\Theta}(\xi) &= -\xi\rfloor\tilde{\mathcal{L}} + \Omega(\xi) + \xi\rfloor\mathrm{d}\Lambda + \mathrm{d}(\xi\rfloor\Lambda) \\ &= -\xi\rfloor\mathcal{L} + \Omega(\xi) + \mathrm{d}(\xi\rfloor\Lambda) = \Theta(\xi) + \mathrm{d}(\xi\rfloor\Lambda). \end{aligned} \tag{4.16} $$

Inserting the decomposition $\Theta(\xi) = \mathcal{S}(\xi) + \mathrm{d}Q(\xi)$, we obtain

$$ \tilde{\Theta}(\xi) = \mathcal{S}(\xi) + \mathrm{d}[Q(\xi) + \xi\rfloor\Lambda]. \tag{4.17} $$

Thus, the total derivative shift of the Lagrangian preserves the algebraic part of the Noether current, whereas it induces a shift of the Noether charge:

$$ \tilde{Q}(\xi) = Q(\xi) + \xi\rfloor\Lambda. \tag{4.18} $$

## 5. Examples

### *5.1. A p-form field*

Consider a Maxwell-type Lagrangian for a $p$-form field $A$ given on an $n$-dimensional manifold

$$ \mathcal{L} = -\tfrac{1}{2} F \wedge {*F} \tag{5.1} $$

where $F = \mathrm{d}A$ is the field strength and $*$ is the Hodge operator.

Variation of the Lagrangian takes the form

$$ \delta\mathcal{L} = -\tfrac{1}{2}\delta F \wedge {*F} - \tfrac{1}{2} F \wedge \delta * F. \tag{5.2} $$

Now, we have to calculate the variation of the form $\delta * F$. For a well-defined notion of the energy–momentum current, the variation $\delta\vartheta^a$ of the coframe field has to be taken into account together with the variation $\delta A$ of the $p$-form field. In this case the variational operator does not commute with the Hodge operator, $\delta * \neq *\delta$. We will use here the master formula [26], which, in the case of (pseudo-)orthonormal coframe, takes the form

$$(\delta * - * \delta)F = \delta\vartheta^a \wedge (e_a \rfloor * F) - *[\delta\vartheta^a \wedge (e_a \rfloor F)]. \tag{5.3}$$

Hence,

$$F \wedge \delta * F = F \wedge *\delta F + \delta\vartheta^a \wedge [(-1)^{p+1} F \wedge (e_a \rfloor * F) - (e_a \rfloor F) \wedge *F] \tag{5.4}$$

and (5.2) reads

$$\delta\mathcal{L} = \delta F \wedge *F + \tfrac{1}{2}\delta\vartheta^a \wedge ((e_a \rfloor F) \wedge *F + (-1)^p F \wedge (e_a \rfloor * F)). \tag{5.5}$$

Thus, the field momentum (2.2) takes the value $\pi = *F$, while the current of the field $A$ vanishes, $\sigma = 0$. Hence, the field equation reads

$$\mathrm{d} * F = 0. \tag{5.6}$$

Using (5.5), we obtain the energy–momentum current as

$$\Sigma_a = \tfrac{1}{2}((e_a \rfloor F) \wedge *F + (-1)^p F \wedge (e_a \rfloor * F)) \tag{5.7}$$

and the Noether charge as

$$Q(e_a) = (e_a \rfloor A) \wedge *F. \tag{5.8}$$

For the Maxwell field ($p = 1$) on a 4D manifold, equation (5.7) gives the correct result.

The trace of the energy–momentum tensor, corresponding to the current (5.7), is proportional to the $n$-form $\vartheta^a \wedge \Sigma_a$. Calculate

$$\vartheta^a \wedge \Sigma_a = -\tfrac{1}{2}(n - 2p - 2)F \wedge *F. \tag{5.9}$$

Thus the corresponding energy–momentum tensor is traceless if and only if $n = 2(p+1)$, i.e., in the case when the strength $F$ is a middle form (for even $n$).

The antisymmetric part of the energy–momentum tensor is proportional to the $(n-2)$-form $e^a \rfloor \Sigma_a$, see, for instance, [32]. Calculating this expression, we obtain

$$e^a \rfloor \Sigma_a = 0. \tag{5.10}$$

Thus, the energy–momentum tensor of the $p$-form field is symmetric for an arbitrary value of the degree $p$.

### 5.2. *Vacuum GR in coframe representation*

Let a differential manifold $M$ be endowed with a *pseudo-orthonormal coframe* field $\vartheta^a$. It means that the metric on $M$ can be represented as

$$g = \eta_{ab}\vartheta^a \otimes \vartheta^b \tag{5.11}$$

where $\eta_{ab} = \mathrm{diag}(-1, 1, \ldots, 1)$. Consequently the Hodge map, which depends on the metric tensor, acts on the basis forms as follows,

$$*(\vartheta^{a_1} \wedge \cdots \wedge \vartheta^{a_q}) = \epsilon^{a_1\cdots a_q\cdots a_n}\vartheta_{a_{q+1}} \wedge \cdots \wedge \vartheta_{a_n} \tag{5.12}$$

where the indices are lowered accordingly to $\vartheta_a := \eta_{ab}\vartheta^b$.

The Einstein–Hilbert Lagrangian corresponds to the coframe Lagrangian [28, 30] as

$$\mathcal{L} = \tfrac{1}{2}[(\mathcal{C}_a \wedge \vartheta^b) \wedge *(\mathcal{C}_b \wedge \vartheta^a) - 2(\mathcal{C}_a \wedge \vartheta^a) \wedge *(\mathcal{C}_b \wedge \vartheta^b)] + \mathrm{d}\Lambda \tag{5.13}$$

where $\mathcal{C}^a := \mathrm{d}\vartheta^a$ and

$$\Lambda = \vartheta^a \wedge *\mathcal{C}_a. \tag{5.14}$$

Using the notation

$$\mathcal{F}^a := \mathcal{C}^a - 2e^a\rfloor(\vartheta^m \wedge \mathcal{C}_m) - \vartheta^a \wedge (e_m\rfloor\mathcal{C}^m) \tag{5.15}$$

the Lagrangian can be written in a compact form as

$$\mathcal{L} = \tfrac{1}{2}\mathcal{C}_a \wedge *\mathcal{F}^a + \mathrm{d}\Lambda. \tag{5.16}$$

Its variation can be written [31, 32] as

$$\delta\mathcal{L} = \delta C^a \wedge *\mathcal{F}_a + \delta\vartheta^a \wedge (e_a\rfloor\mathcal{L} - (e_a\rfloor\mathcal{C}^b) \wedge *\mathcal{F}_b) + \mathrm{d}(\delta\Lambda). \tag{5.17}$$

Thus we identify the field momentum and the conserved current, respectively

$$\Pi^a = *\mathcal{F}_a \qquad \Sigma^a = e_a\rfloor\mathcal{L} - (e_a\rfloor\mathcal{C}^b) \wedge *\mathcal{F}_b. \tag{5.18}$$

Consequently, the field equation is

$$\mathrm{d} * \mathcal{F}^a = \Sigma^a. \tag{5.19}$$

The canonical Noether charge is shifted accordingly to

$$Q(e_a) = *\mathcal{F}_a + e_a\rfloor\Lambda. \tag{5.20}$$

The Lagrangian (5.9) is invariant under two different groups of symmetries:

(i) The (pseudo-)group of diffeomorphism transformations of the manifold which is equivalent to the set of coordinate transformations. Such an invariance is usually referred to as general covariance. All the quantities introduced above (including the conserved current!) are manifestly invariant under these transformations.

(ii) The group of local (pointwise) transformations of the coframe. The Lagrangian preserves its form if we replace the coframe by

$$\vartheta^a \to A^a{}_b(x)\vartheta^b \quad \text{where for all } x \quad A^a{}_b(x) \in SO(1,3). \tag{5.21}$$

Due to the well-known theorem of the calculus of variations, the field equation (5.19) is invariant under these transformations. However, the separation of this equation to the exact form on the left-hand side and the conserved current on the right-hand side is not invariant. Thus also the diffeomorphism invariant conserved current $\Sigma_a$ is not invariant under the local 'internal' transformations (5.21). Certainly, this result corresponds to the known fact that every covariant expression constructed from the first-order derivatives of the metric is trivial. Actually, in view of the complete group of invariance transformation of the Lagrangian, the conserved current (5.18) is only a type of a pseudo-tensor.

### *5.3. Metric-free electrodynamics*

In the axiomatic approach to classical electrodynamics [33–35], spacetime is considered as a four-dimensional differentiable manifold without any additional geometrical structure (metric or connection).

The *first axiom* of the electric charge conservation $\mathrm{d}J = 0$ yields the field equation

$$\mathrm{d}J = 0 \quad \Longrightarrow \quad \mathrm{d}H = J. \tag{5.22}$$

The *second axiom* postulates the existence of the Lorentz force density

$$f_\alpha = (e_\alpha\rfloor F) \wedge J. \tag{5.23}$$

The *third axiom* requires the magnetic flux conservation

$$\mathrm{d}F = 0 \quad \Longrightarrow \quad F = \mathrm{d}A. \tag{5.24}$$

Here $J$ is the electric current density 3-form, $F = (E, B)$ is the untwisted 2-form of the electromagnetic field strength, $H = (\mathcal{H}, \mathcal{D})$ is the twisted 2-form of the electromagnetic excitation, and $e_a$ is a frame field on the manifold. A fairly detailed account, including the conventions and references to the literature, can be found in [34].

To complete the formulation, a relation between $H$ and $F$ is required, namely the constitutive law. This constitutive law is postulated to be *local and linear* ('linear electrodynamics'). The relation between the two forms is established by an odd *constitutive tensor* $\kappa$, which maps even 2-forms to odd 2-forms and vice versa. Namely, $H = \kappa(F)$. We want to find a componentwise representation of $\kappa$. Introduce a coframe field $\vartheta^a$, which is dual to the frame field $e_a$ appearing in (5.23). Thus, $\vartheta^{ab} = \vartheta^a \wedge \vartheta^b$ is a basis for even 2-forms. Using the Levi-Civita pseudo-tensor $\epsilon_{abcd}$ a volume element $\epsilon$ can be defined by the coframe as $\epsilon = \epsilon_{abcd}\vartheta^{abcd}/4!$. Thus, $\epsilon_{ab} = e_a \rfloor e_b \rfloor \epsilon$ is the basis for odd 2-forms. Accordingly, in the components,

$$F = \tfrac{1}{2} F_{ab}\vartheta^{ab} \qquad H = \tfrac{1}{2} H^{ab}\epsilon_{ab}. \tag{5.25}$$

Linearity of the $\kappa$-map means $\epsilon_{ab} = \kappa(\vartheta^{ab})$. Thus, the tensor representation $\chi^{mnab}$ of the operator $\kappa$ reads

$$\kappa(\vartheta^{ab}) := \tfrac{1}{2}\chi^{mnab}\epsilon_{mn} \tag{5.26}$$

and

$$H = \kappa(F) = \tfrac{1}{2}F_{ab}\kappa(\vartheta^{ab}) = \tfrac{1}{4}F_{ab}\chi^{mnab}\epsilon_{mn} \tag{5.27}$$

or, in components,

$$H^{mn} = \tfrac{1}{2}\chi^{mnab}F_{ab}. \tag{5.28}$$

Consider the 'ordinary' Lagrangian of the free electromagnetic field

$$\mathcal{L} = -\tfrac{1}{2}F \wedge H = -\tfrac{1}{2}F \wedge \kappa(F). \tag{5.29}$$

Rewrite the Lagrangian componentwise as

$$\mathcal{L} = -\tfrac{1}{16}F_{ab}\vartheta^{ab} \wedge F_{pq}\chi^{pqmn}\epsilon_{mn} = \tfrac{1}{8}F_{ab}F_{cd}\chi^{abcd}\epsilon. \tag{5.30}$$

Under the action of the group $GL(4, \mathbb{R})$ the tensor $\chi^{abcd}$ can be irreducibly decomposed into three pieces,

$$\chi^{abcd} = {}^{(1)}\chi^{abcd} + {}^{(2)}\chi^{abcd} + {}^{(3)}\chi^{abcd} \tag{5.31}$$

where

$${}^{(3)}\chi^{abcd} := \chi^{[abcd]} \tag{5.32}$$

$${}^{(2)}\chi^{abcd} := \tfrac{1}{2}(\chi^{abcd} - \chi^{cdab}) \tag{5.33}$$

$${}^{(1)}\chi^{abcd} := \chi^{abcd} - {}^{(2)}\chi^{abcd} - {}^{(3)}\chi^{abcd}. \tag{5.34}$$

The Lagrangian (5.30) has an additional symmetry:

$$\chi^{abcd} = \chi^{cdab}. \tag{5.35}$$

Thus the *skewon part* ${}^{(2)}\chi^{abcd}$ does not contribute to this Lagrangian.

We are looking for the energy–momentum current of the electromagnetic fields $F$ and $H$. Due to the linearity of $\kappa$, the constitutive tensor depends only on the non-electromagnetic (geometric) variables. So, its variation will not give a contribution to the energy–momentum current of the electromagnetic field $\Sigma_a$. We will show now that, in order to obtain the true expression for $\Sigma_a$, it is enough to take into account only the variation of the potential $A$ and of the coframe $\vartheta^a$.

We have

$$\delta\mathcal{L} = -\tfrac{1}{2}\delta(F \wedge \kappa F). \tag{5.36}$$

Using the symmetry (5.35), we rewrite the right-hand side as

$$\delta\mathcal{L} = \tfrac{1}{4}\delta(F_{ab})H^{ab}\epsilon + \tfrac{1}{8}F_{ab}H^{ab}\delta(\epsilon). \tag{5.37}$$

Observe now, that the variations on right-hand side of (5.37) do not depend on the odd operator used. Thus, analogously to the standard Maxwell theory and to the $p$-form field expression (5.5), we may write

$$\delta\mathcal{L} = \delta F \wedge H + \delta\vartheta^a \wedge \Sigma_a \tag{5.38}$$

where $\Sigma_a = \partial\mathcal{L}/\partial\vartheta^a$ is the Hilbert energy–momentum current:

$$\Sigma_a = \tfrac{1}{2}\left[(e_a \rfloor F) \wedge H - F \wedge (e_a \rfloor H)\right]. \tag{5.39}$$

Using the commutativity of the operators $\delta$ and d and extracting the total derivative, we obtain the variation of the Lagrangian (5.29) in the form

$$\delta\mathcal{L} = \mathrm{d}(\delta A \wedge H) + \delta A \wedge \mathrm{d}H + \delta\vartheta^a \wedge \Sigma_a. \tag{5.40}$$

In accordance with the non-dynamical coframe procedure (see section 2), we extract from this relation only one field equation $\mathrm{d}H = 0$, which is in accordance with (5.22) providing $J = 0$. For the field that satisfies this field equation, the variation (5.40) reads

$$\delta\mathcal{L} = \mathrm{d}(\delta A \wedge H) + \delta\vartheta^a \wedge \Sigma_a. \tag{5.41}$$

Consider now the variations generated by diffeomorphism invariance of the Lagrangian:

$$\delta A = \xi \rfloor \mathrm{d}A + \mathrm{d}(\xi \rfloor A) \tag{5.42}$$

$$\delta\vartheta^a = \xi \rfloor \mathrm{d}\vartheta^a + \mathrm{d}(\xi \rfloor \vartheta^a) \tag{5.43}$$

$$\delta\mathcal{L} = \mathrm{d}(\xi \rfloor \mathcal{L}). \tag{5.44}$$

We take for non-dynamical coframe $\delta\vartheta^a = 0$. Consequently, relation (5.41) becomes

$$\mathrm{d}\Theta(\xi) + \mathrm{d}(\xi \rfloor \vartheta^a) \wedge \Sigma_a = 0 \tag{5.45}$$

where

$$\Theta(\xi) = -\xi \rfloor \mathcal{L} + [\xi \rfloor \mathrm{d}A + \mathrm{d}(\xi \rfloor A)] \wedge H. \tag{5.46}$$

Extracting the total derivative, we may write it as

$$\Theta(\xi) = \mathcal{S}(\xi) + \mathrm{d}Q(\xi) \tag{5.47}$$

with

$$Q(\xi) = \xi \rfloor A \wedge H \tag{5.48}$$

$$\mathcal{S}(\xi) = -\xi \rfloor \mathcal{L} + \xi \rfloor \mathrm{d}A \wedge H - \xi \rfloor A \wedge \mathrm{d}H. \tag{5.49}$$

On shell $\mathrm{d}H = 0$. Thus (5.45) becomes

$$\mathrm{d}\mathcal{S}(\xi) + \mathrm{d}(\xi \rfloor \vartheta^a) \wedge \Sigma_a = 0 \tag{5.50}$$

with

$$\mathcal{S}(\xi) = -\xi \rfloor \mathcal{L} + \xi \rfloor F \wedge H. \tag{5.51}$$

The first cascade equation (2.33) yields the conservation law

$$\mathrm{d}\mathcal{S}(e_a) = 0 \tag{5.52}$$

whereas the second cascade equation turns out to be an identity $\mathcal{S}(e_a) = -\Sigma_a$. Thus we have derived the conservation law for the Hilbert current

$$\Sigma_a = e_a\rfloor\mathcal{L} - e_a\rfloor F \wedge H \tag{5.53}$$

which is shown to be associated with the diffeomorphism invariance of the Lagrangian.

## 6. Outlook of main results

Let us summarize the results of our consideration. We have examined the Noether procedure for a diffeomorphism invariant Lagrangian in three different ways:

(1) We considered the variation of the Lagrangian, taking into account only the field itself and not involving any information on the geometry of the manifold. In this way, we have obtained a conserved current which identically vanishes on shell.
(2) The second consideration was based on a non-dynamical coframe. Corresponding cascade equations turned out to describe the equivalence between canonical and coframe (Hilbert) current. In this way, we derived a conserved current which, on shell, does not vanish identically.
(3) We have considered a system of two dynamical fields: a $p$-form field $\psi$ and a coframe field $\vartheta^a$. The Noether procedure is consistent in this case. The vanishing canonical current of the system represents a relation between the Noether and the Hilbert currents. Consequently, the conservation law for the canonical current of the $p$-form field yields the conservation law for the Hilbert currents of this field. In this way the Hilbert current is related to the diffeomorphism invariance of the Lagrangian. Consequently, it obtains the status of *the* energy–momentum current. Moreover, the Hilbert current of the system is shown to be the source of the coframe field. We also derive an expression for the Noether charge.

The main result of our consideration is that in order to have a complete and non-contradictory Noether current, it is necessary to involve the geometry of the manifold, such as the coframe field, in the variational procedure.

This result may be correlated with Dicke's analysis [27] which shows that the gravity field is unique in having a Lagrangian describing an 'interaction only with itself'. All other viable Lagrangians have to involve the metric. Thus, in fact, they describe an interaction with the gravitational field.

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